particularly promoted by laminarlike turbulent wake disturbances where streamwise turbulence intensities approach unity as a result of the unstable nature of wake flows.[6,7] Predictions and measurements also show that effects of particle fluxes on PDFs of u and v are small; this behavior generally agrees with numerous observations that the form of the PDFs of velocity fluctuations in turbulent flows is generally independent of the rate of dissipation of the turbulence. Finally, the agreement between measurements and predictions in Figs. 2 and 3 is reasonably good, with predictions correctly representing the effect of particle sizes and fluxes on the PDFs of u and v, particularly in the region where the square of the arguments of the PDFs are smaller than four and present experimental uncertainties are small. The main discrepancies between predictions and measurements are observed in regions where velocities are significantly larger or smaller than the most probable values. These regions correspond to rather small values of the PDFs of u and v, however, and the discrepancies can be largely attributed to both the small values of the PDFs and the sampling limitations of the present measurements (unfeasible orders of magnitude increases in testing times would be required to avoid this limitation).

Summary

In summary, the overall properties of the continuous-phase turbulence generated by uniform fluxes of monodisperse spherical particles moving at near terminal velocities were studied. It was found that many properties of these flows—including relative turbulence intensities, anisotropies, and PDFs—could be predicted reasonably well based on volume-averaged contributions of the conditional averages of these properties within the laminarlike wake disturbance and turbulent interwake regions of the flows. The predictions show that most of the unusual properties of these flows, compared to conventional homogeneous turbulent flows,[4–8] come about as a result of effects of mean velocities in the particle wakes, which cannot be separated from the turbulence (barring some type of wake discrimination system) because the arrival of particle wakes is random.

Acknowledgments

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Associate Editor

Apparently First Closed-Form Solution for Frequency of Beam with Rotational Spring

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Nomenclature

A_j	=	functions defined in Eqs. (10–14)
a_j	=	inertial coefficients
b_I	=	stiffness coefficients
$D(\xi)$	=	flexural stiffness
G_1	=	coefficient defined in Eq. (19)
G_2	=	coefficient defined in Eq. (36)
k_1	=	rotational stiffness
L	=	length
m	=	order of polynomial in Eq. (2)
$R(\xi)$	=	inertial coefficient
$W(\xi)$	=	displacement
x	=	axial coordinate
α_j	=	coefficients defined in Eq. (3)
β	=	nondimensional elastic constant
ξ	=	x/L nondimensional axial coordinate
ω	=	natural frequency

Introduction

CLOSED-FORM solutions for natural frequencies of inhomogeneous beams with pinned supports were derived by Candan and Elishakoff.[1] This Note follows that study, to deal with beams supported by an elastic rotational spring, to derive apparently the first closed-form solution in the literature. To illustrate the feasibility of the proposed method, this Note considers the free vibrations of a beam with rotational elastic support at one end with the other end pinned.

Basic Equations

Consider inhomogeneous beams, governed by the differential equation

$$\frac{\mathrm{d}^2}{\mathrm{d}\xi^2}\left(D(\xi)\frac{\mathrm{d}^2 W}{\mathrm{d}\xi^2}\right) - R(\xi)\omega^2 W(\xi) = 0 \tag{1}$$

Consider the inertial coefficient to be an mth-order polynomial function

$$R(\xi) = \sum_{i=0}^{m} a_i \xi^i \tag{2}$$

The beam's displacement is postulated as a fourth-order polynomial function:

$$W(\xi) = \alpha_0 + \alpha_1\xi + \alpha_2\xi^2 + \alpha_3\xi^3 + \xi^4 \tag{3}$$

The boundary conditions read

$$W(0) = 0, \qquad D(0)\frac{\mathrm{d}^2 W}{\mathrm{d}\xi^2} = \left(\frac{k_1}{L}\right)\frac{\mathrm{d}W}{\mathrm{d}\xi}$$
$$W(1) = 0, \qquad D(1)W''(1) = 0 \tag{4}$$

which corresponds to the rotational spring at $\xi = 0$, of stiffness k_1. To have a compatibility of the order in the polynomial expressions,

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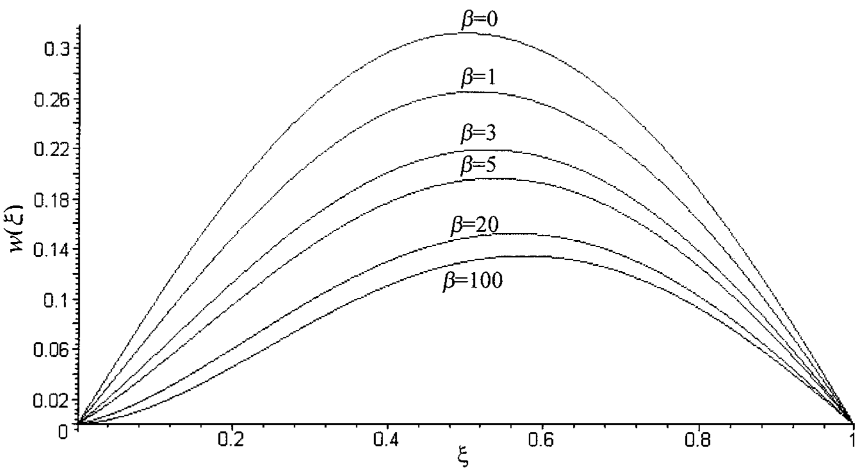

Fig. 1 Mode shapes for different values of coefficient β.

from substitution of Eqs. (2) and (4) into Eq. (1), it is concluded that the stiffness $D(\xi)$ must be represented as a polynomial:

$$D(\xi) = \sum_{i=0}^{m+4} b_i \xi^i \tag{5}$$

Let us consider several particular cases.

Satisfaction of Eq. (4) leads to the following final expression of the postulated mode shape:

$$W(\xi) = 3\frac{b_0 L}{3b_0 L + k_1}\xi + \frac{3}{2}\frac{k_1}{3b_0 L + k_1}\xi^2 - \frac{1}{2}\frac{5k_1 + 12b_0 L}{3b_0 L + k_1}\xi^3 + \xi^4 \tag{6}$$

We introduce the nondimensional elastic constant

$$\beta = k_1 / b_0 L \tag{7}$$

Note that when β tends to zero, Eq. (6) reduces to the mode shape of a pinned-pinned beam:

$$W(\xi) = \xi - 2\xi^3 + \xi^4 \tag{8}$$

whereas when β approaches infinity, Eq. (6) tends to the mode shape of a clamped–pinned beam:

$$W(\xi) = \tfrac{3}{2}\xi^2 - \tfrac{5}{2}\xi^3 + \xi^4 \tag{9}$$

Figure 1 shows the mode shape of the beam for different values of the coefficient β.

Uniform Material Density

Consider first the case, where $m = 0$, corresponding to constant inertial coefficient

$$R(\xi) = a_0 \tag{10}$$

Stiffness in Eq. (5) is then a fourth-order polynomial. Substitution of Eqs. (5–10) into Eq. (1) leads to the following polynomial expressions:

$$A_0 + A_1\xi + A_2\xi^2 + A_3\xi^3 + A_4\xi^4 = 0 \tag{11}$$

with

$$A_0 = 24b_0 + 6\frac{k_1}{3b_0 L + k_1}b_2 - 6\frac{12b_0 L + 5k_1}{3b_0 L + k_1}b_1 \tag{12}$$

$$A_1 = 72b_1 - 18\frac{12b_0 L + 5k_1}{3b_0 L + k_1}b_2 + 18\frac{k_1}{3b_0 L + k_1}b_3 - 3\frac{b_0 L}{3b_0 L + k_1}a_0\omega^2 \tag{13}$$

$$A_2 = 144b_2 - 36\frac{12b_0 L + 5k_1}{3b_0 L + k_1}b_3 + 36\frac{k_1}{3b_0 L + k_1}b_4 - \frac{3}{2}\frac{k_1}{3b_0 L + k_1}a_0\omega^2 \tag{14}$$

$$A_3 = 240b_3 - 60\frac{12b_0 L + 5k_1}{3b_0 L + k_1}b_4 + \frac{1}{2}\frac{12b_0 L + 5k_1}{3b_0 L + k_1}a_0\omega^2 \tag{15}$$

$$A_4 = 360b_4 - a_0\omega^2 \tag{16}$$

Equation (11) is a polynomial function equal to zero for any $\xi \in [0, 1]$. Then, to satisfy this requirement, each coefficient in front of ξ^i for $i = 0, \ldots, 4$ must vanish. Hence, one obtains a system of five equations with six unknowns, namely, b_0, b_1, b_2, b_3, b_4, b_5, and ω^2. Thus, there are an infinite number of closed-form solutions. To derive solutions of this system, we have to express the unknowns in terms of an arbitrary constant. It is convenient to choose the coefficient b_0 as such a constant. Using the nondimensional elastic coefficient β to simplify the analytical expressions, one obtains

$$b_1 = 4(2592 + 1872\beta + 444\beta^2 + 35\beta^3)(3 + \beta)b_0/G_1 \tag{17}$$

$$b_2 = -48(48 + 12\beta - \beta^2)(3 + \beta)^2 b_0/G_1 \tag{18}$$

$$b_3 = -64(3 + \beta)^3(12 + 5\beta)b_0/G_1 \tag{19}$$

$$b_4 = 128(3 + \beta)^4 b_0/G_1 \tag{20}$$

where

$$G_1 = 31{,}104 + 37{,}152\beta + 15{,}696\beta^2 + 2748\beta^3 + 163\beta^4 \tag{21}$$

Because coefficients in G_1 are positive, it takes only positive values for $\beta > 0$.

The natural frequency is expressed from Eq. (16):

$$\begin{aligned} \omega^2 &= 360b_4/a_0 \\ &= 46{,}080(3 + \beta)^4 b_0/a_0 G_1 \end{aligned} \tag{22}$$

Note that without elastic support, when $k = 0$, the stiffness and the natural frequency squared read, respectively,

$$D(\xi) = \left(1 + \tfrac{1}{3}\xi - \tfrac{2}{3}\xi^2 - \tfrac{2}{3}\xi^3 + \tfrac{1}{3}\xi^4\right)b_0$$

$$\omega^2 = 120b_0/a_0 \tag{23}$$

When the elastic coefficient k tends to infinity, the stiffness and the natural frequency squared become

$$D(\xi) = (1/163)(163 + 140\xi - 48\xi^2 - 320\xi^3 + 128\xi^4)b_0$$

$$\omega^2 = (46{,}080/163)b_0/a_0 \tag{24}$$

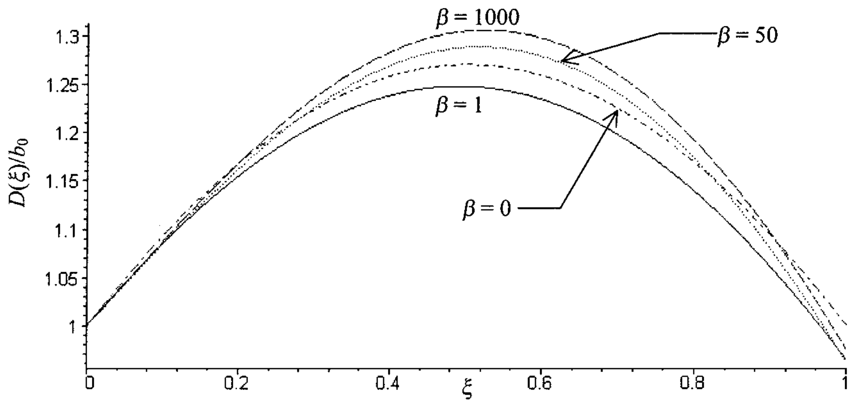

Fig. 2 Variation of the stiffness for constant material density.

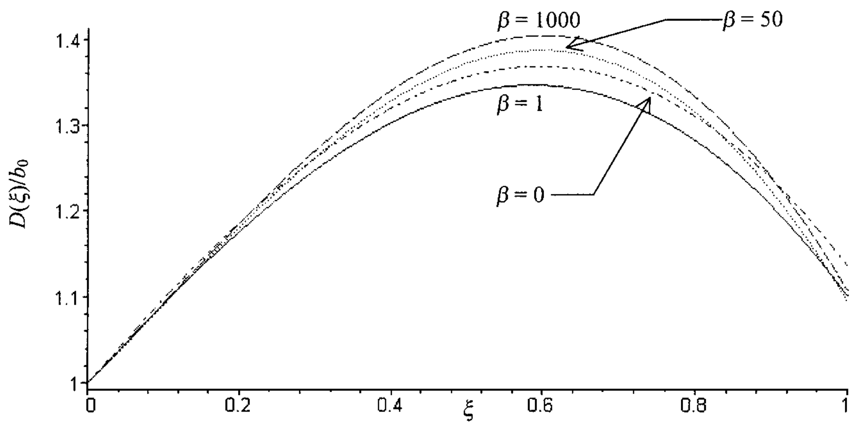

Fig. 3 Variation of the stiffness for constant material density when a_0 and a_1 equal unity.

Figure 2 shows the dependence of $D(\xi)$ vs ξ for various values of the coefficient γ.

Linear Material Density

Consider a beam with a linear material density

$$R(\xi) = a_0 + a_1\xi \tag{25}$$

The stiffness, given by Eq. (5), is a fifth-order polynomial. Thus, the governing equation, with the preceding assumptions, can be rewritten as follows:

$$A_0 + A_1\xi + A_2\xi^2 + A_3\xi^3 + A_4\xi^4 + A_5\xi^5 = 0 \tag{26}$$

with

$$A_0 = 24b_0 + 6\frac{k_1}{3b_0L + k_1}b_2 - 6\frac{12b_0L + 5k_1}{3b_0L + k_1}b_1 \tag{27}$$

$$A_1 = 72b_1 - 18\frac{12b_0L + 5k_1}{3b_0L + k_1}b_2 + 18\frac{k_1}{3b_0L + k_1}b_3 - \frac{3k_1}{3b_0L + k_1}a_0\omega^2 \tag{28}$$

$$A_2 = 144b_2 - 36\frac{12b_0L + 5k_1}{3b_0L + k_1}b_3 + 36\frac{k_1}{3b_0L + k_1}b_4 - \left(\frac{3k_1}{6b_0L + 2k_1}a_0 + \frac{3b_0L}{3b_0L + k_1}a_1\right)\omega^2 \tag{29}$$

$$A_3 = 240b_3 - 60\frac{12b_0L + 5k_1}{3b_0L + k_1}b_4 + 60\frac{k_1}{3b_0L + k_1}b_5 + \left(\frac{12b_0L + 5k_1}{6b_0L + 2k_1}a_0 - \frac{3k_1}{6b_0L + 2k_1}a_1\right)\omega^2 \tag{30}$$

$$A_4 = 360b_4 - 90\frac{12b_0L + 5k_1}{3b_0L + k_1}b_5 - \left(a_0 - \frac{12b_0L + 5k_1}{6b_0L + 2k_1}a_1\right)\omega^2 \tag{31}$$

$$A_5 = 504b_5 - a_1\omega^2 \tag{32}$$

Equation (26) is valid for every ξ within the interval [0;1]. Thus, coefficients A_i, with $i = 0, \ldots, 5$, must vanish. This requirement leads to a system of six equations with seven unknowns. These are the coefficients b_i, with $i = 0, \ldots, 5$, and the natural frequency squared ω^2. We express the unknowns in terms of an arbitrary constant. For convenience, we take the coefficient b_0 as a parameter. Then the other coefficients b_i read

$$b_1 = 4(3 + \beta)\big[(435{,}456 + 459{,}648\beta + 179{,}424\beta^2 + 30{,}744\beta^3 + 1960\beta^4)a_0(176{,}256 + 213{,}408\beta + 91{,}584\beta^2 + 16{,}512\beta^3 + 1047\beta^4)a_1\big]b_0/G_2 \tag{33}$$

$$b_2 = 16(3 + \beta)^2\big[(-24{,}192 - 14{,}112\beta - 1512\beta^2 + 168\beta^3)a_0 \times(14{,}688 + 10{,}368\beta + 2496\beta^2 + 215\beta^3)a_1\big]b_0/G_2 \tag{34}$$

$$b_3 = -64(3 + \beta)^3\big[(2016 + 1512\beta + 280\beta^2)a_0 + (1296 + 384\beta - 7\beta^2)a_1\big]b_0/G_2 \tag{35}$$

$$b_4 = 256(3 + \beta)^4[(84 + 28\beta)a_0 - (108 + 45\beta)a_1]b_0/G_2 \tag{36}$$

$$b_5 = 5120(3 + \beta)^5 a_1 b_0/G_2 \tag{37}$$

where

$$G_2 = (5{,}225{,}472 + 7{,}983{,}360\beta + 4{,}717{,}440\beta^2 + 1{,}340{,}640\beta^3 + 181{,}272\beta^4 + 9158\beta^5)a_0 + (2{,}115{,}072 + 3{,}265{,}920\beta + 1{,}982{,}880\beta^2 + 584{,}640\beta^3 + 82{,}560\beta^4 + 4375\beta^5)a_1 \quad (38)$$

Note that $G_2 = 0$ has only negative roots. It is always positive for physically realizable values of β, namely, $\beta > 0$. Figure 3 shows the dependence of the stiffness vs ξ for different values of the elastic coefficient β. The natural frequency squared reads

$$\omega^2 = 504 b_5 / a_1 = 2{,}580{,}480(3+\beta)^5 b_0 / G_2 \quad (39)$$

Because G_2 is positive, so is the natural frequency for $\beta > 0$.

When the elastic coefficient vanishes, the stiffness and the natural frequency read, respectively,

$$D(\xi) = b_0 + \big[12(435{,}456a_0 + 176{,}256a_1)\xi + 144(-24{,}192a_0 + 14{,}688a_1)\xi^2 - 1728(2016a_0 + 1296a_1)\xi^3 + 20{,}736(84a_0 - 108a_1)\xi^4 + 1{,}244{,}160a_1\xi^5\big] b_0/(5{,}225{,}472a_0 + 2{,}115{,}072a_1)$$

$$\omega^2 = 627{,}056{,}640 b_0/(5{,}225{,}472a_0 + 2{,}115{,}072a_1) \quad (40)$$

When the elastic coefficient tends to the infinity, the stiffness and the natural frequency tend to the following respective expressions:

$$D(\xi) = b_0 + \big[4(1960a_0 + 1047a_1)\xi + 16(168a_0 + 215a_1)\xi^2 - 64(280a_0 - 7a_1)\xi^3 + 256(28a_0 - 45a_1)\xi^4 + 5120a_1\xi^5\big] b_0/(9128a_0 + 4375a_1)$$

$$\omega^2 = 2{,}580{,}480 b_0/(9128a_0 + 4375a_1) \quad (41)$$

Figure 3 shows the variation of the stiffness for various values of β.

In an analogous manner one can determine the closed-form solutions for natural frequencies of beams with parabolic, cubic, quartic, etc., variations of the mass density.

Summary

This study presents a simple closed-form solution for the natural frequency and the mode shape of the inhomogeneous beam with rotational spring. The author is unaware of any other closed-form solution for the homogeneous or inhomogeneous beam with a spring. Analogous treatment can be performed for beams with springs at each end.

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Associate Editor

Prediction of Fundamental Frequencies of Stressed Spring-Hinged Tapered Beams

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Introduction

INTERSTAGES, thrust frames, payload adapters, etc., of rocket/missile structural systems are often configured with beams attached to end rings. The beams are joined to the end rings, and based on the joint configuration the ends of the beams are usually idealized using end rotational constraints, the stiffness parameter value of which is fixed based on the quality of the joint. These beams during their service conditions are subjected to compressive loads. Further, the vibration behavior of these beams, especially the fundamental frequency of prediction, is most important for evaluation of the transmissibility of gravitational acceleration loads due to the flight dynamic environment. To save mass, these beams are often designed as tapered beams, which are near optimum for engineering purposes.

Quick prediction and information about the fundamental frequency of such beams under initial stress is of significant importance during their design phase. The finite element method is widely used to estimate the fundamental frequency of beams with initial stresses.[1–5] End rotational constraints are considered in Refs. 6–10. Though the finite element method is versatile, closed-form solution is very elegant and yields quick results. In this Note, a simple formula is derived to predict the fundamental frequency of stressed tapered beams with end rotational restraints when its stress free frequency is known. It is assumed that the mode shapes for the buckling and free vibration are the same as that of the initially stressed vibration of the tapered beam. A set of numerical results for tapered beams with end rotational restraints having symmetric linear variation in depth and constant breadth are generated for various values of rotational stiffness parameters using the formula. The finite element method (the beam is idealized using 20 tapered elements) is employed to verify these results. The details of the formulation and the numerical results are presented.

Formulation

Consider a beam of length L, moment of inertia $I(x)$, where x is the axial coordinate, E Young's modulus, and ρ mass density with an end concentrated load P (Fig. 1). If the beam is executing harmonic oscillations, then the governing matrix equation of the free vibrations of the initially stressed beam is given by the standard equation

$$[K]\{\delta\} - \lambda[G]\{\delta\} - \lambda_f[M]\{\delta\} = 0 \quad (1)$$

where $[K]$, $[G]$, and $[M]$ are the assembled elastic stiffness matrix with the proper contribution of end rotational spring stiffness, geometric stiffness matrix, and mass matrix, respectively, and λ is the axial compressive load parameter, which is time invariant (defined as $\lambda = PL^2/EI_0$, where I_0 is the reference moment of inertia at the ends of the beam), and λ_f is the frequency parameter (defined as $\lambda_f = \rho A_0 \omega^2 L^4/EI_0$, where ω is the circular frequency, A_0 is the reference cross-sectional area at the ends of the beam), and $\{\delta\}$ is the eigenvector.

From Eq. (1), the degenerate case of the stability equation is given by

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